EXACT HYPERPLANE COVERS FOR SUBSETS OF THE HYPERCUBE

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Abstract. Alon and Füredi (1993) showed that the number of hyperplanes required to cover $\{0,1\}^n \setminus \{0\}$ without covering 0 is n. We initiate the study of such exact hyperplane covers of the hypercube for other subsets of the hypercube. In particular, we provide exact solutions for covering $\{0,1\}^n$ while missing up to four points and give asymptotic bounds in the general case. Several interesting questions are left open.

1. Introduction

A vector $a \in \mathbb{R}^n$ and a scalar $b \in \mathbb{R}$ determine the hyperplane

$$\{x \in \mathbb{R}^n : \langle a, x\rangle = a_1x_1 + \cdots + a_nx_n = b\}$$

in $\mathbb{R}^n$. How many hyperplanes are needed to cover $\{0,1\}^n$? Only two are required; for instance, $\{x : x_1 = 0\}$ and $\{x : x_1 = 1\}$ will do. What happens however if $0 \in \mathbb{R}^n$ is not allowed on any of the hyperplanes? We can 'exactly' cover $\{0,1\}^n \setminus \{0\}$ with n planes: for example, the collections $\{\{x : x_i = 1\} : i \in [n]\}$ or $\{\{x : \sum_{i=1}^n x_i = j\} : j \in [n]\}$ can be used, where $[n] := \{1, 2, \ldots, n\}$. Alon and Füredi [1] showed that in fact n planes are always necessary.

Recently, a variation was studied by Clifton and Huang [4], in which they require that each point from $\{0,1\}^n\setminus\{0\}$ is covered at least k times for some $k \in \mathbb{N}$ (while 0 is never covered). Another natural generalisation is to remove more than just 0 from the set of points we wish to cover exactly. For $B \subseteq \{0,1\}^n$, let $\mathrm{ec}(B)$ denote the *exact cover number* of B, i.e. the minimum number of hyperplanes whose union intersects $\{0,1\}^n$ exactly in B. We will usually write B in the form $\{0,1\}^n \setminus S$ for some subset $S \subseteq \{0,1\}^n$. In particular, the result of Alon and Füredi [1] states that $\mathrm{ec}(\{0,1\}^n \setminus \{0\}) = n$.

We first determine what happens if we remove up to four points.

Theorem 1. *Let $S \subseteq \{0,1\}^n$.*

- *If $|S| \in \{2,3\}$, then $\mathrm{ec}(\{0,1\}^n \setminus S) = n - 1$.*
- *If $|S| = 4$, then $\mathrm{ec}(\{0,1\}^n \setminus S) = n - 1$ if there is a hyperplane Q with $|Q \cap S| = 3$ and $\mathrm{ec}(\{0,1\}^n \setminus S) = n - 2$ otherwise.*

For $n \in \mathbb{N}$ and $k \in [2^n]$, we also introduce the exact cover numbers

$$\mathrm{ec}(n,k) = \max\{\mathrm{ec}(\{0,1\}^n \setminus S) : S \subseteq \{0,1\}^n,\ |S| = k\},$$
$$\mathrm{ec}(n) = \max\{\mathrm{ec}(B) : B \subseteq \{0,1\}^n\}.$$

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We make some progress towards determining the asymptotics of $\mathrm{ec}(n,k)$ and $\mathrm{ec}(n)$ with the following two results.

Theorem 2. *For any positive integer k, $\mathrm{ec}(n,k) = n + O_k(1)$.*

Theorem 3. *For n sufficiently large, $2^{n-2}/n^2 \leq \mathrm{ec}(n) \leq 2^{n+1}/n$.*

The problem of determining the asymptotics of $\mathrm{ec}(n)$ was also suggested by Füredi at Alon's birthday conference.

2. Covering all but up to four points

In this section, we determine $\mathrm{ec}(\{0,1\}^n \setminus S)$ for subsets S of size 2, 3 and 4. For the lower bounds, we use the following result of Alon and Füredi [1].

Theorem 4 (Corollary 1 in [1])**.** *If $n \geq m \geq 1$, then m hyperplanes that do not cover all vertices of $\{0,1\}^n$ miss at least 2^{n-m} vertices.*

For the upper bounds, it suffices to give an explicit construction of a collection of hyperplanes that exactly covers $\{0,1\}^n \setminus S$, for every subset S of size 2, 3 or 4. We split the proof of Theorem 1 into two cases, the case where $|S| \in \{2,3\}$ and the case where $|S| = 4$.

Lemma 5. *Let $S \subseteq \{0,1\}^n$ with $|S| \in \{2,3\}$. Then $\mathrm{ec}(\{0,1\}^n \setminus S) = n-1$.*

Proof. Let $S \subseteq \{0,1\}^n$ with $|S| \in \{2,3\}$. We first prove the lower bound $\mathrm{ec}(\{0,1\}^n \setminus S) \geq n-1$; this follows from applying the case of $m = n-2$ in Theorem 4. Indeed, this shows that any $n-2$ planes that do not cover all of $\{0,1\}^n$ miss at least 4 vertices, and hence a minimum of $n-1$ planes are required to miss 2 or 3 vertices.

For the upper bound, note that we may assume by vertex transitivity that $(0,\ldots,0) \in S$. Consider first the case $|S| = 2$. By relabelling the indices, we may assume the second vector u in S satisfies $\{i \in [n] : u_i = 1\} = \{1,\ldots,\ell\}$ for some $\ell \in \mathbb{N}$. We cover $\{0,1\}^n \setminus S$ by the collection of $n-1$ hyperplanes

$$\{\{x : x_i = 1\} : i \in \{\ell+1,\ldots,n\}\} \cup \{\{x : x_1 + \cdots + x_\ell = j\} : j \in [\ell-1]\},$$

and none of these hyperplanes contain an element from S.

Now consider the case $|S| = 3$. We may assume the second and third vector in S correspond to the subsets $\{1,\ldots,a+b\}$ and $\{1,\ldots,a\} \cup \{a+b+1,\ldots,a+b+c\}$ for some $a,b,c \in \mathbb{Z}_{\geq 0}$ with $a+b \geq 1$ and $c \geq 1$. We first add the $n-(a+b+c)$ planes of the form $\{x : x_i = 1\}$ for $i \in \{a+b+c+1,\ldots,n\}$. For $x \in S$, we have

$$\begin{aligned} &x_1 + \cdots + x_a \in \{0,a\},\\ &x_{a+1} + \cdots + x_{a+b} \in \{0,b\},\\ &x_{a+b+1} + \cdots + x_{a+b+c} \in \{0,c\}. \end{aligned}$$

If $a \geq 1$, we add the $a-1$ planes $\{x : x_1 + \cdots + x_a = i\}$ for $i \in [a-1]$, and we proceed similarly for b and c. The only points of $\{0,1\} \setminus S$ that are yet to be covered satisfy the equations above and also satisfy $x_i = 0$ for $i > a+b+c$.

Suppose first that $a,b \geq 1$. In this case we have added $n-3$ planes so far. The problem has effectively been reduced to covering $\{0,1\}^3$ with three missing points $(0,0,0),(1,1,0)$ and $(1,0,1)$ using 2 planes. Indeed, we may add the following two planes to our collection in order to exactly cover $\{0,1\}^n \setminus S$:

$$\left\{x : \frac{x_1+\cdots+x_a}{a} + \frac{x_{a+1}+\cdots+x_{a+b}}{b} + \frac{x_{a+b+1}+\cdots+x_{a+b+c}}{c} = 1\right\},$$
$$\left\{x : \frac{x_{a+1}+\cdots+x_{a+b}}{b} + \frac{x_{a+b+1}+\cdots+x_{a+b+c}}{c} = 2\right\}.$$

Suppose now that $a=0$ or $b=0$. Since $a+b \geq 1$ and $c \geq 1$, we have used $n-2$ planes so far. If $a=0$, we may add the plane

$$\left\{x : \frac{x_1+\cdots+x_b}{b} + \frac{x_{b+1}+\cdots+x_{b+c}}{c} = 2\right\}$$

and, if $b=0$, we add

$$\left\{x : -\frac{x_1+\cdots+x_a}{a} + \frac{x_{a+1}+\cdots+x_{a+c}}{c} = 1\right\}.$$

In either case, the resulting collection covers $\{0,1\} \setminus S$ without covering any point in S. □

For the case of four missing points, we always need at least $n-2$ planes by Theorem 4. For $n=3$, we may need either 1 or 2 planes. For example, we may exactly cover $\{0,1\}^3 \setminus (\{0\} \times \{0,1\}^2)$ by the single plane $\{x : x_1 = 1\}$, but if S does not lie on a plane then we need two planes. The set $\{0\} \times \{0,1\}^2$ has the special property that there is no plane that covers three of its points without covering the fourth. It turns out this condition is exactly what decides how many planes are required when removing four points.

Lemma 6. *Let $S \subseteq \{0,1\}^n$ with $|S| = 4$. Then $\mathrm{ec}(\{0,1\}^n \setminus S) = n-1$ if there is a hyperplane Q with $|Q \cap S| = 3$ and $\mathrm{ec}(\{0,1\}^n \setminus S) = n-2$ otherwise.*

Proof. We know that $\mathrm{ec}(\{0,1\}^n \setminus S) \geq n-2$ from Theorem 4. If there is a plane P intersecting S in exactly three points, then $\mathrm{ec}(\{0,1\}^n \setminus S) \geq n-1$. Indeed, by vertex transitivity, we may assume that 0 is the point of S uncovered by P. Any exact cover of $\{0,1\}^n \setminus S$ can be extended to an exact cover of $\{0,1\}^n \setminus \{0\}$ by adding the plane P to the collection.

We prove the claimed upper bounds by induction on n, handling the case $n \leq 7$ by computer search. Again, we may assume that $0 \in S$. Let u, v, w denote the other three vectors in S. For any i with $u_i = v_i = w_i = 0$, we can use a plane of the form $\{x : x_i = 1\}$ to reduce the covering problem to one of a lower dimension. (Note that dropping the coordinate i in this case does not change whether three points in S can be covered without covering the fourth.) Hence we may assume by induction that no such i exists.

After possibly permuting coordinates, we assume that $u_i = v_i = w_i = 1$ on the first a coordinates, $u_i = v_i = 1$ and $w_i = 0$ on the b coordinates after that, etcetera, so that our four vectors take the form

$$\begin{pmatrix} 0 \\ u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \tag{1}$$

where each column may be replaced with 0 or more columns of its type. Since $n > 7$, by the pigeonhole principle one of the columns must be repeated at least twice. We will show how to handle the case for which this is the first column (i.e. $a \geq 2$); the other cases are analogous.

Our collection of planes will contain the planes

$$\{\{x : x_1 + \cdots + x_a = i\} : i \in [a-1]\}. \tag{2}$$

The only points x which have yet to be covered have the property that x_i takes the same value in $\{0,1\}$ for all $i \in [a]$. We now proceed similarly to the proof of Lemma 5. Informally, we wish to 'merge' the first a coordinates and then apply the induction hypothesis. For each $s \in S$, we define $\pi(s) = (s_a, \dots, s_n)$. Let $\pi(S) = \{\pi(s) : s \in S\}$. Then $|S| = |\pi(S)| = 4$.

Any hyperplane

$$P = \{y : c_1 y_1 + \cdots + c_{n-a+1} y_{n-a+1} = b\}$$

in $\{0,1\}^{n-a+1}$ can be used to define a hyperplane

$$L(P) = \left\{x : c_1 \frac{x_1 + \cdots + x_a}{a} + c_2 x_{a+1} + \cdots + c_{n-a+1} x_n = b\right\}$$

in $\{0,1\}^n$. For all $x \in \{0,1\}^n$ with $\sum_{i=1}^a x_i \in \{0,a\}$, we find that $\pi(x) \in P$ if and only if $x \in L(P)$. This shows that if $P_1, \dots, P_M$ form an exact cover for $\{0,1\}^{n-a+1} \setminus \pi(S)$, then $L(P_1), \dots, L(P_M)$, together with the planes from (2), form an exact cover for $\{0,1\}^n \setminus S$. This proves

$$\mathrm{ec}(\{0,1\}^n \setminus S) \leq \mathrm{ec}(\{0,1\}^{n-a+1} \setminus \pi(S)) + a - 1.$$

Since there is a plane covering three points in S without covering the fourth if and only if this is the case for $\pi(S)$, we find the claimed upper bounds by induction.

Observe that the proof reduction works also in the case $n \leq 7$ if there are at least two coordinates of the same type in (1). Thus, the computer verification is needed only in the case when each column in (1) appears at most once. The code used to check the small cases is attached to the arXiv submission. □

3. Asymptotics

We first consider the asymptotics of $\mathrm{ec}(n,k)$ when k is held fixed. For the upper bound, we prove the following lemma.

Lemma 7. *For all $k \in \mathbb{N}$ and $n \geq 2^{k-1}$, $\mathrm{ec}(n,k) \leq 1 + \mathrm{ec}(n-1,k)$.*

Proof. Fix $k \in \mathbb{N}$, $n \geq 2^{k-1}$ and a subset $S \subseteq \{0,1\}^n$ of size $|S| = k$. For $i \in [n]$, let $S_{-i} \subseteq \{0,1\}^{n-1}$ be obtained from S by deleting coordinate i from each element of S. We claim that there exists an $i \in [n]$ such that $|S_{-i}| = k$ and

$$\mathrm{ec}(\{0,1\}^n \setminus S) \leq 1 + \mathrm{ec}(\{0,1\}^{n-1} \setminus S_{-i}). \tag{3}$$

The lemma follows immediately from this claim.

By vertex transitivity, we may assume that $0 \in S$. Suppose first that there exists $i \in [n]$ for which $s_i = 0$ for all $s \in S$. Then $|S_{-i}| = k$. From an exact cover for $\{0,1\}^{n-1} \setminus S_{-i}$, we may obtain an exact cover for $\{x \in \{0,1\}^n \setminus S : x_i = 0\}$. Combining with the plane $\{x : x_i = 1\}$, this gives an exact cover for $\{0,1\}^n \setminus S$. This proves (3).

We henceforth assume that $0 \in S$ and that the remaining $k-1$ elements of S cannot all be 0 on the same coordinate. Hence there are at most $2^{k-1}-1$ possible values that $(s_i : s \in S)$ can take for $i \in [n]$. Since $n \geq 2^{k-1}$, by the pigeonhole principle, there must exist coordinates $1 \leq i < j \leq n$ with $s_i = s_j$ for all $s \in S$. This implies that $|S_{-i}| = |S| = k$. We now show (3) is satisfied. After permuting coordinates, we may assume that $(i,j) = (1,2)$. An exact cover for $\{0,1\}^{n-1} \setminus S_{-1}$ is converted to an exact cover for $\{0,1\}^n \setminus S$ as in the proof of Lemma 6: any plane of the form

$$P = \{y : c_1 y_1 + \cdots + c_{n-1} y_{n-1} = b\}$$

is converted to

$$L(P) = \left\{x : c_1 \frac{x_1 + x_2}{2} + c_2 x_3 + \cdots + c_{n-1} x_n = b\right\},$$

and we add the plane $\{x : x_1 + x_2 = 1\}$ to the adjusted collection. □

It is now easy to prove to that $\mathrm{ec}(n,k) = n + O_k(1)$.

Proof of Theorem 2. Let $k \in \mathbb{N}$. We prove that for all $n \geq 2^k$,

$$n - \log_2(k) \leq \mathrm{ec}(n,k) \leq n - 2^k + \mathrm{ec}(2^k, k).$$

The upper bound is vacuous for $n = 2^k$ and follows from $n - 2^k$ applications of Lemma 7 for $n > 2^k$. The lower bound follows from Theorem 4: if $n - \ell$ planes cover all but k vertices, then $k \geq 2^\ell$, and hence $n - \ell \geq n - \log_2(k)$. (In fact, this shows $\mathrm{ec}(\{0,1\}^n \setminus S) \geq n - \log_2(k)$ for each subset $S \subseteq \{0,1\}^n$ of size k.) □

We now turn to the problem when $|S|$ is not held fixed. We use two auxiliary lemmas.

For the lower bound, we use a random argument for which we need to know the approximate number of intersection patterns of the hypercube. An *intersection pattern* of $\{0,1\}^n$ is a non-empty subset $P \subseteq \{0,1\}^n$ for which there exists a hyperplane H with $H \cap \{0,1\}^n = P$.

Lemma 8. *$\{0,1\}^n$ has at most 2^{n^2} possible intersection patterns.*

Proof. We will associate each intersection pattern with a unique element from $(\{0,1\}^n)^n$. Let $P \subseteq \{0,1\}^n$ be an intersection pattern with $P = H \cap \{0,1\}^n$ for H a hyperplane. Then $|P| < 2^n$.

Choose $x \in P$ for which $\sum_{i=1}^n x_i 2^i$ is minimal. Let $\oplus$ denote coordinate-wise addition modulo 2 and write $x \oplus P = \{x \oplus p : p \in P\} \subseteq \{0,1\}^n$. Note that $0 \in x \oplus P$ since $x \in P$, and that $x \oplus P$ is the intersection of a linear subspace of dimension $n-1$ with $\{0,1\}^n$. (The linear subspace can be obtained from H by a series of reflections.) We greedily find $0 \leq k \leq n-1$ linearly independent vectors $v_1, \ldots, v_k \in x \oplus P$ whose linear span intersects $\{0,1\}^n$ in $x \oplus P$. We label P with the n-tuple $(x, v_1, \ldots, v_k, 0, \ldots, 0)$, where we added $n-1-k$ copies of the vector 0 at the end of the tuple. This associates each intersection pattern to a unique element from $(\{0,1\}^n)^n$. □

The above proof is rather crude, but in fact not far from the truth: the number of possible intersection patterns is $2^{(1+o(1))n^2}$ (see e.g. [3]).

We also use an auxiliary result for the upper bound. The *total domination number* of a graph G is the minimum cardinality of a subset $D \subseteq V(G)$ such that each $v \in V(G)$ has a neighbour in D.

Lemma 9 (Theorem 5.2 in [5])**.** *The total domination number of the hypercube is at most $2^{n+1}/n$ for n sufficiently large.*

We are now ready to prove $2^{n-2}/n^2 \leq \text{ec}(n) \leq 2^{n+1}/n$ (for n sufficiently large).

Proof of Theorem 3. For the lower bound, we need to give a subset $B \subseteq \{0,1\}^n$ that is difficult to cover exactly. We will find a subset S for which all large intersection patterns have a non-empty intersection with S. This means that to cover $\{0,1\}^n \setminus S$, we can only use small planes. We take a subset $S \subseteq \{0,1\}^n$ at random by including each point independently with probability $1/2$.

For any fixed intersection pattern P, the probability that it is disjoint from our random set S is $\left(\frac{1}{2}\right)^{|P|}$. By Lemma 8, for n sufficiently large, there are at most 2^{n^2} possible intersection patterns. Hence, by the union bound, the probability that there is an intersection pattern which has at least $2n^2$ elements and does not intersect with S, is at most $2^{n^2}\left(\frac{1}{2}\right)^{2n^2} = o(1)$. With probability at least $1/2$, our random set S has at most 2^{n-1} points. Hence for n sufficiently large, there exists a subset S of size 2^{n-1} that 'hits' all intersection patterns of size at least $2n^2$. Any exact cover for $\{0,1\}^n \setminus S$ consists entirely of planes whose intersection pattern has size $\leq 2n^2$, and hence needs at least $|\{0,1\}^n \setminus S|/2n^2 = 2^{n-2}/n^2$ planes.

We now prove the upper bound. The Hamming distance on $\{0,1\}^n$ is given by $d(x,y) = \sum_{i=1}^n |x_i - y_i|$. A Hamming sphere around a point $x \in \{0,1\}^n$ is given by $S(x) = \{y \in \{0,1\}^n : d(x,y) = 1\}$. We claim that any subset of a Hamming sphere is an intersection pattern. Since the cube is vertex-transitive, it suffices to prove our claim for $S(0)$. The plane $\{x : \sum_{i=1}^n x_i = 1\}$ intersects $\{0,1\}^n$ in $S(0)$. Intersecting that plane with planes of the form $\{x : x_j = 0\}$ gives a lower-dimensional affine subspace, and we can construct such a subspace which intersects $S(0)$ in any subset we desire. In order to turn the affine subspace into a hyperplane with the same intersection pattern, we may add directions that do not yield new points in the hypercube by adding directions such as $(1, \pi, 0, \dots, 0)$. This proves each subset of a Hamming sphere is an intersection pattern.

The hypercube has total domination number at most $2^{n+1}/n$ by Lemma 9. Hence we can find a subset D of the cube such that each vertex has a neighbour in D. In particular, there are $M \leq 2^{n+1}/n$ Hamming spheres centered on the vertices in D that cover the cube. For any $B \subseteq \{0,1\}^n$, we write $B = B_1 \cup \cdots \cup B_M$ such that each B_i is covered by at least one of the Hamming spheres. This means that each B_i is a intersection pattern, and hence we may cover B exactly using M hyperplanes. This gives the desired exact cover of B with at most $2^{n+1}/n$ hyperplanes. □

We finish this section by observing what happens if the original Alon-Füredi problem is restricted to a single layer.

Proposition 10. *Let $n \in \mathbb{N}$ and $i \in \{0, \dots, n\}$. Let B be obtained by removing a single point from the ith layer $\{x \in \{0,1\}^n : |x| = i\}$. Then $\mathrm{ec}(B) = \min\{i, n-i\}$.*

Proof. We may assume that $i \leq n/2$ and that $b = (1, \dots, 1, 0, \dots, 0)$ is the missing point. The upper bound follows by taking the planes

$$\{\{x : x_1 + \cdots + x_i = j\} : j \in \{0, \dots, i-1\}\}.$$

For the lower bound, we claim that we may find a cube of dimension i within the ith layer for which b plays the role of the origin. Indeed, consider the map

$$\iota : \{0,1\}^i \to \{0,1\}^n : x \mapsto (1 - x_1, 1 - x_2, \dots, 1 - x_i, 0, \dots, 0, x_i, x_{i-1}, \dots, x_1).$$

That is, we view the point b as the origin and take the directions of the form $(-1, 0, \dots, 0, 1)$, $(0, -1, 0, \dots, 0, 1, 0)$, etcetera, as the axes of the cube. Now $(B \setminus \{b\}) \cap \iota(\{0,1\}^i) = \iota(\{0,1\}^i \setminus \{0\})$, and hence we may convert any cover for $B \setminus \{b\}$ to a cover for $\{0,1\}^i \setminus \{0\}$. The lower bound follows from the result of Alon and Füredi [1]. □

4. Conclusion

Based on the fact that $\mathrm{ec}(n,k) \leq n$ for $k = 1, 2, 3, 4$, one might hope to prove that in fact $\mathrm{ec}(n,k) \leq n$. However, this is not true in general by Theorem 3. A natural question is then whether this will be true for n sufficiently large when k is fixed. It must then somehow be the case that if we place the 'same subset' in a space with more dimensions, the exact cover problem becomes easier. We conjecture this is not the case in a very strong way.

Conjecture 1. *For any $S \subseteq \{0,1\}^r$ and $n \in \mathbb{N}$ with $n \geq r$,*

$$\mathrm{ec}(\{0,1\}^n \setminus (S \times \{0\}^{n-r})) = \mathrm{ec}(\{0,1\}^r \setminus S) + n - r.$$

The upper bound is immediate and the lower bound holds for $|S| = 1, 2, 3, 4$ by the results of the previous section.

The conjecture can be used to give an inductive proof of the Alon-Füredi result on covering $\{0,1\}^n \setminus \{0\}$. Moreover, assuming the conjecture is true, for every constant $K > 0$ there is a value of $k \in \mathbb{N}$ for which $\mathrm{ec}(n,k) \geq n + K$ for all n sufficiently large. Indeed, by Theorem 3 we may find r sufficiently large and $S \subseteq \{0,1\}^r$ with

$$\mathrm{ec}(\{0,1\}^r \setminus S) \geq 2^{r-2}/r^2 \geq r + K.$$

Setting $k = |S|$, the conjecture would then imply that for any $n \geq r$

$$\mathrm{ec}(n,k) \geq \mathrm{ec}(\{0,1\}^n \setminus (S \times \{0\}^{n-r})) \geq \mathrm{ec}(\{0,1\}^r \setminus S) + n - r \geq n + K.$$

One approach to improving the lower bound in Theorem 3 is to try to prove that, for some $\varepsilon \in (0,1)$, the number of planes containing $n^{1+\varepsilon}$ points is $O(2^{n^{1+\varepsilon}})$. Unfortunately, this is false: there are $2^{(1+o(1))n^2}$ possible intersection patterns of size at least n^2. This can be seen by considering intersection patterns of the form $\{0,1\}^{\log(n^2)} \times B$ for $B \subseteq \{0,1\}^{n-\log(n^2)}$. (If B is a non-empty intersection pattern, then $\{0,1\}^{\log(n^2)} \times B$ is an intersection pattern containing n^2 points.) On the other hand, by taking every other layer we may intersect each 'axis-aligned subcube' of the form $\{0,1\}^a \times \{x\}$, ensuring that no such intersection pattern can be used in a cover. However, there is a more general type of subcube to consider.

We say a subset $A \subseteq \{0,1\}^n$ of size $|A| = 2^d$ forms a d-dimensional *subcube* if there are vectors $c, v_1, \ldots, v_d \in \mathbb{R}^n$ such that

$$A = \{c + a_1v_1 + \cdots + a_dv_d : a_1, \ldots, a_d \in \{0,1\}^d\}.$$

A solution to the following problem might help improve either the upper or lower bound of Theorem 3.

Problem 1. *Fix $n, d \in \mathbb{N}$. What is the smallest cardinality of a subset $S \subseteq \{0,1\}^n$ for which $A \cap S \neq \emptyset$ for all d-dimensional subcubes $A \subseteq \{0,1\}^n$?*

This is of a similar flavour to a problem proposed by Alon, Krech and Szábo [2], who asked instead for the asymptotics of the above problem when the cubes have to be axis-aligned. A d-dimensional axis-aligned subcube is of the form $\{0,1\}^d \times \{x\}$ after permuting coordinates. Let $g(n,d)$ denote the minimal cardinality of a subset that hits all such d-dimensional subcubes in $\{0,1\}^n$ and let $c_d^0 = \lim_{n\to\infty} \frac{g(n,d)}{2^n}$. The best-known bounds for this problem are

$$\frac{\log(d)}{2^{d+2}} \leq c_d^0 \leq \frac{1}{d+1}$$

from [2].

Finally, we remark that we have already seen these subcubes come up in Lemma 6 as well: the sets $S \subseteq \{0,1\}^n$ of size 4 with $\mathrm{ec}(\{0,1\}^n \setminus S) = n - 2$ are exactly the 2-dimensional subcubes.

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